A RANK INEQUALITY FOR THE TATE CONJECTURE OVER GLOBAL FUNCTION FIELDS

CHRISTOPHER LYONS

We present an observation of D. Ramakrishnan concerning the Tate Conjecture for varieties over a global function field (i.e., the function field of a smooth projecture curve over a finite field), which was pointed out during a lecture given at the AIM's workshop on the Tate Conjecture in July 2007. The result is perhaps "known to the experts," but we record it here, as it does not appear to be in print elsewhere. We use the global Langlands correspondence for the groups GL_n over global function fields, proved by L. Lafforgue [Laf], along with an analytic result of H. Jacquet and J. Shalika [JS] on automorphic L-functions for GL_n. Specifically, we use these to show (see Theorem 2.1 below) that, for a prime $\ell \neq \operatorname{char} k$, the dimension of the subspace spanned by the rational cycles of codimension m on our variety in its $2m$-th ℓ-adic cohomology group (the so-called algebraic rank) is bounded above by the order of the pole at $s = m + 1$ of the associated L-function (the so-called analytic rank). The interest in this result lies in the fact that, with the exception of some special instances like certain Shimura varieties and abelian varieties which are potentially CM type, the analogous result for varieties over number fields is still unknown in general, even for the case of divisors ($m = 1$).

1. Preliminaries

Tate's original article [Tat1] serves as a good reference for this section, and also gives insight into the motivation behind the conjectures. The similar case of varieties over $\mathbb{Q}$, which has the additional advantage that singular cohomology and Hodge theory can be brought to bear on the problem, is discussed in §1 of [Ram].

Let X be a smooth, projective, geometrically connected variety over a global function field k. Let $\mathbb{F}_q$ denote the constant field of k and $\bar{k}$ its separable closure. Fix a prime $\ell \neq \operatorname{char} k$. For an integer $0 \leq m \leq \dim X$, write

$$V_\ell = H^{2m}_{\text{ét}}(X \times_k \bar{k}, \mathbb{Q}_\ell)$$

for the $2m$-th ℓ-adic cohomology group, which is a finite-dimensional vector space over $\mathbb{Q}_\ell$. The natural action of $\Gamma_k := \mathrm{Gal}(\bar{k}/k)$ on $\bar{k}$ gives an action of Γ_k on $X \times_k \bar{k}$, which in turn gives rise to a continuous linear action of Γ_k on V_ℓ. Thus we get a continuous representation $\rho_\ell \colon \Gamma_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell)$. Moreover, for almost every place

Date: November 2008.

v of k (i.e., for all but a finite number), ρ_ℓ is *unramified* at v, in the sense that the inertia subgroup I_v of any decomposition group D_v for v is in the kernel of ρ_ℓ.

To this representation ρ_ℓ of Γ_k can be associated an L-function $L(\rho_\ell, s)$; we will not need the full L-function, but rather the incomplete form $L^S(\rho_\ell, s)$, where S is any finite set of places containing those where either ρ_ℓ is ramified or X has bad reduction. By definition,

$$L^S(\rho_\ell, s) = \prod_{v \notin S} L_v(\rho_\ell, s),$$

where

$$L_v(\rho_\ell, s) = \det(1 - q_v^{-s}\rho_\ell(Fr_v))^{-1}$$

for any $v \notin S$. Here Fr_v is the *geometric* Frobenius conjugacy class of v in Γ_k and q_v is the residue cardinality of v. Then by the proof of the Weil Conjectures [Del2], we have $L_v(\rho_\ell, s) = Z_v(q_v^{-s})$, where $Z_v(T)$ is a polynomial with coefficients in $\mathbb{Z}$ which factors as

$$Z_v(T) = \prod_{i=1}^{b}(1 - \alpha_{i,v}T),$$

where $b = \dim_{\mathbb{Q}_\ell} V_\ell$ and each $\alpha_{i,v}$ has absolute value q_v^m under any complex embedding. (Note that the $\alpha_{i,v}$ are the eigenvalues of $\rho_\ell(Fr_v)$.) It follows that the Euler product $L^S(\rho_\ell, s)$ converges absolutely for $\mathrm{Re}(s) > m + 1$, and in fact uniformly on compact subsets, giving a holomorphic function in this half-plane.

Now let C^m denote group of cycles of codimension m on X, which is the free abelian group generated by closed irreducible subvarieties of codimension m on $X \times_k \overline{k}$. Let

$$V_\ell(m) := V_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(m);$$

here we set

$$\mathbb{Q}_\ell(1) := \left(\varprojlim_j \mu_{\ell^j}\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

with the action of Γ_k given by its action on each μ_{ℓ^j}, the group of ℓ^jth roots of unity of in $\overline{k}$, and then we take $\mathbb{Q}_\ell(m) := \mathbb{Q}_\ell(1)^{\otimes m}$. (One calls $V_\ell(m)$ the mth *Tate twist* of V_ℓ.) One can show (see [Mil], VI.9) the existence of a canonical cycle class map

$$\mathrm{cl}_m \colon C^m \to V_\ell(m).$$

There is a natural Γ_k-action on C^m coming from that on $X \times_k \overline{k}$, and it turns out that cl_m is a morphism of Γ_k-modules (i.e., is a Γ_k-equivariant map). This means that a cycle in $(C^m)^{\Gamma_k}$ maps into $V_\ell(m)^{\Gamma_k}$.

Define the following quantities:

$$
\begin{aligned}
r^{(m)}_{\mathrm{alg},k} &= \dim_{\mathbb{Q}_\ell}\Big[\mathrm{cl}_m\Big((C^m)^{\Gamma_k}\Big)\otimes\mathbb{Q}_\ell\Big],\\
r^{(m)}_{\ell,k} &= \dim_{\mathbb{Q}_\ell} V_\ell(m)^{\Gamma_k},\\
r^{(m)}_{\mathrm{an},k} &= -\mathrm{ord}_{s=m+1} L^S(\rho_\ell, s).
\end{aligned}
\tag{1a}
$$

(If $L^S(\rho_\ell, s)$ is known to have meromorphic continuation to the point $s = m+1$, this last quantity makes sense as the order of pole at $s = m+1$; otherwise we take it to be the unique integer a, if it exists, such that

$$\lim_{s\to m+1}(s-m-1)^a L^S(\rho_\ell, s)$$

is finite and nonzero. Also note that $r^{(m)}_{\mathrm{an},k}$ is independent of our choice of S by Deligne's proof of the Weil Conjectures, as long as S satisfies the aforementioned conditions.) The first and last quantities are referred to as the *algebraic* and *analytic* ranks, respectively. The Γ_k-equivariance of cl_m above gives that

$$r^{(m)}_{\mathrm{alg},k} \le r^{(m)}_{\ell,k}.$$

J. Tate's conjecture [Tat1] is that, in fact, all three quantities in (1a) are equal.

2. Statement of main theorem and a consequence

In §5 we will show

Theorem 2.1. *For a smooth, projective, geometrically connected variety X over a global function field k, we have*

$$r^{(m)}_{\ell,k} = r^{(m)}_{\mathrm{an},k},$$

and thus

$$r^{(m)}_{\mathrm{alg},k} \le r^{(m)}_{\mathrm{an},k},$$

for any $0 \le m \le \dim X$.

Let us discuss a consequence of this result. For any finite extension L of k, let

$$r^{(m)}_{\mathrm{an},L} = -\mathrm{ord}_{s=m+1} L^S(\rho_{\ell|_{\Gamma_L}}, s).$$

(Note we are abusing notation slightly, since S should really be replaced with a finite set S_L of places of L containing those lying above the places in S, but this is unimportant.) Similarly, define

$$r^{(m)}_{\ell,L} = \dim_{\mathbb{Q}_\ell} V_\ell(m)^{\Gamma_L}$$

for the action of Γ_L via $\rho_{\ell|_{\Gamma_L}}$. We should mention that this notation is consistent, in the following sense: looking at the variety $X_L := X \times_k L$ over L with its continuous action of Γ_L on

$$H^{2m}_{\text{ét}}(X_L \times_L \bar{L}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(m)$$

and the associated L-function, $r^{(m)}_{\ell,L}$ as just defined is the dimension of the classes fixed by Γ_L, and $r^{(m)}_{\mathrm{an},L}$ as just defined is equal the analytic rank. Hence, because Theorem 2.1 is also true for X_L over L, we have $r^{(m)}_{\mathrm{an},L} = r^{(m)}_{\ell,L}$.

The corollary to Theorem 2.1 we wish to discuss is that the numbers

$$\left\{ r^{(m)}_{\mathrm{an},L} \;\middle|\; L/k \text{ finite} \right\}$$

exhibit a certain naturality as L varies which, given their analytic definitions, is not apparent at first sight. Let us explain this naturality and how it follows from the theorem.

First note that, if M is a finite extension of L, so that $\Gamma_M \subseteq \Gamma_L$, then we have

$$r^{(m)}_{\mathrm{an},L} = r^{(m)}_{\ell,L} \leq r^{(m)}_{\ell,M} = r^{(m)}_{\mathrm{an},M}.$$

Next define the subspace $T^{(m)}_\ell$ of *Tate classes* in $V_\ell(m)$ to be those classes whose stabilizer is an open subgroup of Γ_k. Of course we have $V_\ell(m)^{\Gamma_L} \subseteq T^{(m)}_\ell$ since Γ_L is open, so

$$r^{(m)}_{\mathrm{an},L} = r^{(m)}_{\ell,L} \leq \dim_{\mathbb{Q}_\ell} T^{(m)}_\ell.$$

Moreover, the definition of $T^{(m)}_\ell$ and finite-dimensionality of $V_\ell(m)$ imply there is some smallest finite extension $k^{(m)}_\ell/k$ such that $\Gamma_{k^{(m)}_\ell}$ acts trivially on $T^{(m)}_\ell$. Thus

$$r^{(m)}_{\mathrm{an},L} = r^{(m)}_{\ell,L} = \dim_{\mathbb{Q}_\ell} T^{(m)}_\ell$$

whenever L contains k_ℓ. In conclusion, the integers $r^{(m)}_{\mathrm{an},L}$

- are nonnegative,
- have an ordering which is governed by the ordering of inclusion of finite extensions,
- are bounded above,
- achieve this upper bound exactly when $k^{(m)}_\ell \subseteq L$.

We note that, in the number field case, the analogues of both Theorem 2.1 and the corollary just discussed are unknown. It would be interesting if, in lieu of proving Theorem 2.1 when k is a number field, one could still establish these naturality properties of the analytically-defined collection $\{r^{(m)}_{\ell,L}\}$.

3. Automorphic representations of $\mathrm{GL}_n(\mathbb{A}_k)$ and their L-functions

Our strategy in proving that $r^{(m)}_{\ell,k} = r^{(m)}_{\mathrm{an},k}$ is to use Lafforgue's result that the representation ρ_ℓ is *modular*; that is to say, there is an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_k)$ whose L-function has the same analytic behavior as that of ρ_ℓ. This is fortuitous, since the analytic behavior of automorphic L-functions is a priori much better understood than that of L-functions of Galois representations such as ρ_ℓ. For this reason, we take this to section to briefly recall facts about L-functions of cuspidal automorphic representations. We refer the reader to §1.2 of [Ram] or §1.1 of [Lau] for a more thorough introduction.

With k still being a global function field, let $\mathbb{A}_k$ denote its ring of adeles, and let ω denote a *unitary* idele class character of k. We define a space of functions

$$L^2(\omega) := L^2(\mathrm{GL}_n(k)Z(\mathbb{A}_k)\backslash \mathrm{GL}_n(\mathbb{A}_k), \omega),$$

where $Z(\mathbb{A}_k) \simeq \mathbb{A}_k^\times$ denotes the center of $\mathrm{GL}_n(\mathbb{A}_k)$, as the (classes of) measurable functions $\phi\colon \mathrm{GL}_n(\mathbb{A}_k) \to \mathbb{C}$ which satisfy

- $\phi(\gamma g z) = \omega(z)\phi(g)$ for all $\gamma \in \mathrm{GL}_n(k)$, $g \in \mathrm{GL}_n(\mathbb{A}_k)$, and $z \in Z(\mathbb{A}_k)$,
- $\int_{\mathrm{GL}_n(k)Z(\mathbb{A}_k)\backslash \mathrm{GL}_n(\mathbb{A}_k)} |\phi(g)|^2 \, dg < \infty$;

note that the second condition makes sense, since the first condition and ω being unitary allow $|\phi|$ to descend to a function on $\mathrm{GL}_n(k)Z(\mathbb{A}_k)\backslash \mathrm{GL}_n(\mathbb{A}_k)$. There is a subspace $L^2_{\mathrm{cusp}}(\omega)$ of $L^2(\omega)$ of those functions ϕ satisfying the following condition: if U is the unipotent radical of any standard parabolic subgroup of GL_n, then we have

$$\int_{U(k)\backslash U(\mathbb{A}_k)} \phi(ug)du = 0$$

for almost all $g \in \mathrm{GL}_n(\mathbb{A}_k)$. This subspace $L^2_{\mathrm{cusp}}(\omega)$ is referred to as the space of *cusp forms* on $\mathrm{GL}_n(\mathbb{A}_k)$ of *central character* ω.

We have a left action of $\mathrm{GL}_n(\mathbb{A}_k)$ on $L^2(\omega)$ by right translations (that is, by the action $(h \cdot \varphi)(g) := \varphi(gh)$ for $h \in \mathrm{GL}_n(\mathbb{A}_k)$). This action happens to preserve $L^2_{\mathrm{cusp}}(\omega)$, and thus $L^2_{\mathrm{cusp}}(\omega)$ yields a complex representation of $\mathrm{GL}_n(\mathbb{A}_k)$. This representation comes with a number of desirable properties: in particular, we have a semisimple decomposition

$$L^2_{\mathrm{cusp}}(\omega) \simeq \widehat{\bigoplus}_\pi V_\pi^{m_\pi},$$

where (π, V_π) runs over a system of representatives for isomorphism classes of irreducible *admissible* complex representations of $\mathrm{GL}_n(\mathbb{A}_k)$. Furthermore, the *multiplicity one theorem for* GL_n of Shalika says that, for any such π, we have either $m_\pi = 1$ or $m_\pi = 0$. We define a *cuspidal automorphic representation* of $\mathrm{GL}_n(\mathbb{A}_k)$ (or simply a *cuspidal representation*) with central character ω to be any component (π, V_π) of this direct sum for which $m_\pi = 1$.

Now let ω be an arbitrary idele class character of $\mathbb{A}_k^\times/k^\times$, which is not necessarily unitary. Let $\|\cdot\|_{\mathbb{A}_k}$ denote the adelic norm on $\mathbb{A}_k$. Then there is a unique $t \in \mathbb{R}$ and a unique unitary idele class character ω_0 such that

$$\omega = \omega_0 \, \|\cdot\|_{\mathbb{A}_k}^t .$$

One may take the definition of a cuspidal representation π of $\mathrm{GL}_n(\mathbb{A}_k)$ with central character ω to be one of the form

$$\pi := \pi' \otimes (\|\cdot\|_{\mathbb{A}_k}^t \circ \det),$$

where π' is a cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_k)$ with central character ω_0, as defined above. From now on, we use the term "cuspidal representation" in this sense, with no restriction on the central character unless otherwise specified.

For each cuspidal representation π, it turns out that $\pi \simeq \bigotimes'_v \pi_v$, which is a restricted tensor product that runs over the places v of k. Each factor (π_v, V_{π_v}) is a complex representation of $\mathrm{GL}_n(k_v)$ which is irreducible and admissible. Let $\mathcal{O}_v$ be the ring of integers in k_v and let $K_v = \mathrm{GL}_n(\mathcal{O}_v)$. We say that π_v is *unramified* if $V_{\pi_v}^{K_v}$ is nontrivial. For cuspidal π, one knows π_v is unramified for almost every v.

Inspired by a theorem of I. Satake, R. Langlands attached to any unramified irreducible admissible complex representation π_v of $\mathrm{GL}_n(k_v)$ an unordered n-tuple $\{\beta_{1,v}, \beta_{2,v}, \ldots, \beta_{n,v}\}$ of nonzero complex numbers. These numbers, called the *Langlands parameters* (or just the *parameters*) of π_v, determine π_v up to isomorphism. Hence a cuspidal representation π determines such an n-tuple for all v at which π is unramified.

One fact needed below is that, if λ is an idele class character and π' is a cuspidal representation, then the representation

$$\pi := \pi' \otimes (\lambda \circ \det)$$

is also cuspidal. Furthermore, if v is a place such that λ_v is unramified and π'_v is unramified with parameters $\{\beta_{j,v}\}$, then π_v is also unramified and has parameters $\{\beta_{j,v}\lambda(\varpi_v)\}$, where ϖ_v is a uniformizer for k_v.

Given π, it is known ([JS],[JPSS]) that knowledge of the parameters of π_v for almost every unramified v is enough to determine π up to isomorphism, as long as π has unitary central character (which is always true after an appropriate twist by $\|\cdot\|^t_{\mathbb{A}_k}$, $t \in \mathbb{R}$):

Theorem 3.1 ("strong mulitiplicity one"; Jacquet, I. Piatetski-Shapiro, Shalika)**.** *Suppose π_1 and π_2 are two cuspidal representations, both with unitary central character, satisfying $\pi_{1,v} \simeq \pi_{2,v}$ for all v outside some finite set S of places of k. Then $\pi_1 \simeq \pi_2$.*

If π_v is unramified, define

$$L_v(\pi, s) = \left[(1 - \beta_{1,v} q_v^{-s}) \cdots (1 - \beta_{n,v} q_v^{-s})\right]^{-1}$$

and let

$$L^S(\pi, s) = \prod_{v \notin S} L_v(\pi, s)$$

be the incomplete L-function associated to π, where S is a finite set of places containing those at which π is ramified. Then in [JS] (see Propositions 3.3 and 3.6) the following result is proved:

Theorem 3.2 (Jacquet, Shalika)**.** *Suppose that π has unitary central character. Then $L^S(\pi, s)$ is holomorphic for* $\mathrm{Re}(s) > 0$ *if π is not an idele class character of the form $\|\cdot\|^{it}_{\mathbb{A}}$, $t \in \mathbb{R}$.*

On the other hand, when $\pi = \|\cdot\|_{\mathbb{A}}^{it}$, so that $n = 1$ and π_v is unramified everywhere, we have $\beta_{1,v} = q_v^{-it}$ for all v. Hence in this case, $L^S(\pi, s)$ is the translated Dedekind zeta function $\zeta_k(s + it)$ of k (divided by a finite number of Euler factors if $S \neq \emptyset$), which is holomorphic in $\mathbb{C}$ except for a simple pole at $s = 1 - it$. In particular, we have

Corollary 3.3. *Suppose that π has unitary central character. Then*

$$-\operatorname{ord}_{s=1} L^S(\pi, s) = \begin{cases} 1 & \textit{if } \pi \textit{ trivial} \\ 0 & \textit{if } \pi \textit{ nontrivial} \end{cases}. \tag{3a}$$

4. ℓ-adic representations and the Langlands correspondence over k for GL_n

Lafforgue's result pairs each irreducible ℓ-adic Galois representation with a cuspidal representation. We will describe the objects on the first side more explicitly, and then describe the correspondence. The survey [Lau] is a good reference for this material, notably §1.2 and §1.3. We then give an easy extension of this result.

For any $n \geq 1$, we will define an *n-dimensional ℓ-adic representation* of Γ_k to be a continuous homomorphism $\sigma_\ell \colon \Gamma_k \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(M)$ for some finite-dimensional vector space M over $\overline{\mathbb{Q}}_\ell$. Let $\mathcal{G}'_n$ denote a system of representatives for the isomorphism classes of *irreducible* n-dimensional ℓ-adic representations σ_ℓ of Γ_k which satisfy the following three additional properties:

(i) There is a basis of M such that, when using this basis to identify $\mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(M)$ with $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, one has $\sigma_\ell(\Gamma_k) \subseteq \mathrm{GL}_n(E)$ for some finite extension $E \subseteq \overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$.

(ii) There are only a finite number of places v of k at which σ_ℓ is ramified, in the sense described in §1.

(iii) The character $\det \sigma_\ell$ is of finite order.

At this point, we fix once and for all an isomorphism $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. To any such σ_ℓ we can assign an incomplete L-function $L^S(\sigma_\ell, s)$, for a finite set S containing the ramified places of σ_ℓ, in exactly the same manner as in §1: for $v \notin S$, set

$$L_v(\sigma_\ell, s) = \det(1 - q_v^{-s}\sigma_\ell(Fr_v))^{-1} \tag{4a}$$

and then set

$$L^S(\sigma_\ell, s) = \prod_{v \notin S} L_v(\sigma_\ell, s).$$

Thanks to the isomorphism ι, we view this as a complex L-function.

Let $\mathcal{A}'_n$ denote a system of representatives for the isomorphism classes of cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_k)$ with *finite order* central character. Then the following *global Langlands correspondence for* GL_n was proved for the case $n = 2$ by V. Drinfeld [Dri1],[Dri2] and later extended to all cases $n > 2$ by Lafforgue [Laf] (with the case $n = 1$ following from class field theory for k):

Theorem 4.1 (Lafforgue)**.** *There is a unique bijection $\mathcal{G}'_n \to \mathcal{A}'_n$, $\sigma_\ell \mapsto \pi$, such that for almost every place v at which σ_ℓ and π are unramified,*

$$L_v(\sigma_\ell, s) = L_v(\pi, s).$$

We now discuss how to extend this theorem to the case where the "finite order" restrictions are removed from the definitions of $\mathcal{G}'_n$ and $\mathcal{A}'_n$. This extension is something which is presumably well-known to experts, but does not seem to be written down. The key ingredient is the description of unramified (Galois and idele class) characters for k given by class field theory in the function field setting.

Define $\mathcal{A}_n$ to be a system of representatives for the isomorphism classes of cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_k)$ (with no restriction on the central character). Also let $\mathcal{G}_n$ be defined exactly as $\mathcal{G}'_n$ above, but without condition (iii), and define $L_v(\sigma_\ell, s)$ using (4a) if $\sigma_\ell \in \mathcal{G}_n$ is unramified at v. Then we have the following:

Corollary 4.2. *There is a unique bijection $\mathcal{G}_n \to \mathcal{A}_n$, $\sigma_\ell \mapsto \pi$, such that for almost every place v at which σ_ℓ and π are unramified,*

$$L_v(\sigma_\ell, s) = L_v(\pi, s). \tag{4b}$$

(We note that it is this bijection which is stated in the papers of Drinfeld. The finite-order assumptions are only present in Lafforgue's work, and are not serious obstacles, as this corollary demonstrates.)

Before getting to the proof of this corollary, we need the following result:

Lemma 4.3. *Let E be a finite extension of $\mathbb{Q}_\ell$ and let $\chi\colon \Gamma_k \to E^\times$ be a continuous character. Then there is a finite power of χ which is unramified everywhere.*

We remark that this statement is false for number fields, due mainly to the presence of archimedean places. (See §6 for more details.)

Proof of 4.3. By compactness of Γ_k, we may assume χ takes values in $\mathcal{O}_E^\times \subseteq E^\times$ by changing basis ([Ser], p.1). We have an isomorphism

$$\mathcal{O}_E^\times \simeq \mu_E \times \mathcal{O}_E,$$

where μ_E is the group of roots of unity in E. If ℓ^r is the cardinality of the residue field of E, then μ_E is cyclic of order $\ell^r - 1$, while $\mathcal{O}_E$ is a pro-ℓ group.

Now let v be any place of k. By local class field theory, if I_v is the inertia subgroup of any decomposition group $D_v \subseteq \Gamma_k$ of v, then the image of I_v in the abelianization Γ_k^{ab} of Γ_k is the product of a finite cyclic group and a pro-p group, where $p = \mathrm{char}\, k \neq \ell$. Since χ factors through Γ_k^{ab}, this forces $\chi(I_v) \subseteq \mu_E \times \{0\}$ and shows that $\chi^{\ell^r - 1}$ is unramified. □

Proof of 4.2. Pick any $\sigma_\ell \in \mathcal{G}_n$, and suppose, by (i), that σ_ℓ takes values in $\mathrm{GL}_n(E)$ for a finite extension E of $\mathbb{Q}_\ell$.

The character $\chi = \det \sigma_\ell$ is continuous and takes values in $E^\times$, so by the lemma we pick $w \in \mathbb{Z}$ such that χ^w is unramified. By global class field theory for k (see

[AT], p.76), this means that χ^w factors through $\mathrm{Gal}(k\overline{\mathbb{F}}_q/k) \simeq \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \simeq \hat{\mathbb{Z}}$ (recall that $\mathbb{F}_q$ is the constant field of k) and is completely determined by the image of $1 \in \hat{\mathbb{Z}}$. Denoting this element as $\chi^w(1)$ by abuse of notation, we choose some $z \in \overline{\mathbb{Q}}_\ell$ such that $z^{wn} = \chi^w(1)$.

Let $\lambda_\ell \colon \Gamma_k \to E(z)^\times$ be the unique unramified character such that, again by abuse of notation, $\lambda_\ell(1) = z$ and thus $\lambda_\ell^{wn} = \chi^w$. By global class field theory, λ_ℓ corresponds to an unramified idele class character $\lambda\colon \mathbb{A}_k^\times/k^\times \to \mathbb{C}^\times$, in the sense that $\lambda_\ell(Fr_v) = \lambda(\varpi_v)$ for all v, where ϖ_v is a uniformizer of k_v. (Note that this is the opposite convention of that in [AT], since Fr_v is the geometric Frobenius. Also recall we have identified $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$ via the fixed isomorphism ι.)

Since σ_ℓ is unramified almost everywhere, $\sigma_\ell \otimes \lambda_\ell^{-1}$ is a continuous representation of Γ_k which also is unramified almost everywhere and that takes values in $\mathrm{GL}_n(E(z))$. Furthermore,

$$(\det(\sigma_\ell \otimes \lambda_\ell^{-1}))^w = (\chi\lambda_\ell^{-n})^w = 1,$$

i.e., the determinant of $\sigma_\ell \otimes \lambda_\ell^{-1}$ has finite order. Thus Theorem 4.1 gives a unique cuspidal representation π', with central character of finite order, such that

$$L_v(\pi', s) = L_v(\sigma_\ell \otimes \lambda_\ell^{-1}, s) \tag{4c}$$

for almost all v.

Let S be a finite set of places containing those for which (4c) does not hold, as well as the ramified places of π' and σ_ℓ. For $v \notin S$, (4c) means that the parameters $\{\beta_{j,v}\}$ of π'_v coincide with the eigenvalues of

$$(\sigma_\ell \otimes \lambda_\ell^{-1})(Fr_v) = \sigma_\ell(Fr_v)\lambda_\ell(Fr_v)^{-1}.$$

This implies that the parameters $\{\beta_{j,v}\lambda(\varpi_v)\}$ of the cuspidal representation $\pi = \pi' \otimes(\lambda \circ \det)$ coincide with the set of eigenvalues of $\sigma_\ell(Fr_v)$, and therefore that

$$L_v(\pi, s) = L_v(\sigma_\ell, s)$$

for $v \notin S$.

Let us denote this construction of π from σ_ℓ as $r_n\colon \sigma_\ell \mapsto \pi$. We have verified that almost all local L-factors of σ_ℓ and π agree, as required in the statement of the corollary. We now verify that r_n satisfies the other necessary properties.

r_n is well-defined: The only potential ambiguity in our construction is the choice of z such that $z^{wn} = \chi^w(1)$, and hence the choice of λ_ℓ. Suppose that $\tilde{\lambda}_\ell$ were another valid choice, corresponding to the idele class character $\tilde{\lambda}$. Then the representations π' and $\tilde{\pi}'$ associated to $\sigma_\ell \otimes \lambda_\ell^{-1}$ and $\sigma_\ell \otimes \tilde{\lambda}_\ell^{-1}$, respectively, may indeed differ. However, the representations $\pi' \otimes (\lambda \circ \det)$ and $\tilde{\pi}' \otimes (\tilde{\lambda} \circ \det)$ will be the same, as one can verify by comparing their parameters and using the strong multiplicity one theorem. Hence π is defined unambiguously.

r_n is injective: Here one uses the fact that knowledge of almost every local factor $L_v(\sigma_\ell, s)$ determines σ_ℓ up to isomorphism, essentially by Chebotarev density (see the theorem on p.I-10 of [Ser], which applies to all global fields).

r_n is surjective: Pick $\pi \in \mathcal{A}_n$ with central character ω. Then

$$\omega = \omega_f \, \|\cdot\|_{\mathbb{A}_k}^y$$

for a finite order character ω_f and some $y \in \mathbb{C}$. Indeed, because k is a function field, this follows from the fact that the kernel of $\|\cdot\|_{\mathbb{A}_k}$ is compact and countable, and so its complex characters are all of finite order, as well as the fact that the image of $\|\cdot\|_{\mathbb{A}_k}$ is isomorphic to $\mathbb{Z}$. Let $\lambda = \|\cdot\|_{\mathbb{A}_k}^y$, which is an unramified idele class character, and let $\lambda_\ell \colon \Gamma_k \to \overline{\mathbb{Q}}_\ell^\times$ be the corresponding unramified ℓ-adic character, in the sense described above. Then $\pi \otimes (\lambda \circ \det)^{-1} \in \mathcal{A}'_n$ corresponds, by the theorem, to a representation $\sigma'_\ell \in \mathcal{G}'_n$, and one checks that this implies

$$L_v(\pi, s) = L_v(\sigma'_\ell \otimes \lambda_\ell, s)$$

for almost every v. Thus π corresponds to $\sigma_\ell := \sigma'_\ell \otimes \lambda_\ell$.

r_n is the unique bijection satisfying (4b) for almost all v: Were there another bijection with this property, we would wind up with two nonisomorphic cuspidal representations π_1, π_2 whose parameters match at almost every place v. Thus, for some place v, we have an isomorphism $\pi_{1,v} \simeq \pi_{2,v}$ of unramified representations of $\mathrm{GL}_n(k_v)$. This implies the central characters of $\pi_{1,v}$ and $\pi_{2,v}$ are both equal to $|\cdot|_v^z$ for some $z \in \mathbb{C}$; here, $|\cdot|_v$ is the normalized absolute value on k_v. It follows that if

$$\pi'_i := \pi_i \otimes (\|\cdot\|_{\mathbb{A}_k}^{-\operatorname{Re} z} \circ \det)$$

for $i = 1, 2$, then each π'_i has unitary central character and $\pi'_{1,v} \simeq \pi'_{2,v}$ for almost every v. By the strong multiplicity one theorem, this gives $\pi'_1 \simeq \pi'_2$, and hence $\pi_1 \simeq \pi_2$, a contradiction. So r_n must be the unique bijection with the given property. □

5. Proof of Theorem 2.1

Recall the setup and notation in §1. We have now reviewed the tools needed to prove our main result:

Theorem 2.1. *For a smooth, projective, geometrically connected variety X over a global function field k, we have*

$$r_{\ell,k}^{(m)} = r_{\mathrm{an},k}^{(m)},$$

and thus

$$r_{\mathrm{alg},k}^{(m)} \le r_{\mathrm{an},k}^{(m)},$$

for any $0 \le m \le \dim X$.

Proof. Let $\rho_\ell(m)\colon \Gamma_k \to \mathrm{Aut}_{\mathbb{Q}_\ell} V_\ell(m)$ denote the m-th Tate twist of ρ_ℓ.

The semisimplification of the extension of $\rho_\ell(m)$ to an action on $V_\ell(m) \otimes \overline{\mathbb{Q}}_\ell$ is a direct sum of irreducible $\overline{\mathbb{Q}}_\ell$-representations. An easy exercise shows the existence of a finite extension $E/\mathbb{Q}_\ell$ over which this semisimple decomposition is defined. In other words, we have

$$(V_\ell(m) \otimes E)^{\mathrm{ss}} = \bigoplus_i M_i$$

where each M_i is an E-vector space such that Γ_k acts irreducibly on $M_i \otimes \overline{\mathbb{Q}}_\ell$ (and hence irreducibly on M_i) via the extension of $\rho_\ell(m)$.

Let $\rho_i\colon \Gamma_k \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(M_i \otimes \overline{\mathbb{Q}}_\ell)$ denote the irreducible $\overline{\mathbb{Q}}_\ell$-representation defined by $\rho_\ell(m)$. Recall from §1 that, because $\rho_\ell(m)$ arises from the cohomology of X, it is unramified at almost every place of k; thus ρ_i inherits this property as well. Hence, because ρ_i is defined over a finite extension $E/\mathbb{Q}_\ell$ as just remarked, it follows that $\rho_i \in \mathcal{G}_{n_i}$ in the notation of §4, where $n_i = \dim M_i$. By Corollary 4.2, there is a unique cuspidal representation $\pi_i \in \mathcal{A}_{n_i}$ such that

$$L_v(\pi_i, s) = L_v(\rho_i, s) \tag{5a}$$

for almost every v.

Recall from §1 that the eigenvalues of almost every $\rho_\ell(Fr_v)$ are algebraic and have absolute value q_v^m for any complex embedding. Since the action of Γ_k on the $\mathbb{Q}_\ell(m)$ is unramified everywhere, and Fr_v acts on it by q_v^{-m}, it follows that the eigenvalues of almost every $\rho_\ell(m)(Fr_v)$ have absolute value 1 in every complex embedding. Thus the same is true of the eigenvalues of almost every $\rho_i(Fr_v)$. Following the proof of 4.2, this implies that the central character of π_i is unitary.

For the rest of the proof, fix a finite set of places S of k satisfying the following: If $v \notin S$, then $\rho_\ell(m)$ (and hence each ρ_i) is unramified at v, X has good reduction at v, and (5a) holds for all i.

The knowledge of almost every local L-factor $L_v(\pi_i, s)$ equivalent to knowing the parameters of almost every unramified local representation $\pi_{i,v}$ and so, by the strong multiplicity one theorem (applicable because π_i has unitary central character), this knowledge determines π_i up to isomorphism. On the other hand, Chebotarev density (see [Ser], loc. cit.) shows that knowledge of almost every local L-factor $L_v(\rho_i, s)$ determines ρ_i up to isomorphism. Hence the equalities in (5a), which hold for all $v \notin S$, show that π_i is trivial (i.e., $L_v(\pi_i, s) = 1 - q_v^{-s}$ for all v) if and only if ρ_i is trivial (i.e., $L_v(\rho_i, s) = 1 - q_v^{-s}$ for all v). So by Corollary 3.3 we get

$$-\mathrm{ord}_{s=1} L^S(\rho_i, s) = \begin{cases} 1 & \text{if } \rho_i \text{ trivial} \\ 0 & \text{if } \rho_i \text{ nontrivial} \end{cases}. \tag{5b}$$

Next we note that for $v \notin S$, the local L-factor $L_v(\rho_\ell(m), s)$ is the same whether we regard Γ_k as acting on $V_\ell(m)$ or on $V_\ell(m) \otimes \overline{\mathbb{Q}}_\ell$. Thus for $v \notin S$ we have

$$L_v(\rho_\ell(m), s) = \prod_i L_v(\rho_i, s),$$

and hence

$$L^S(\rho_\ell(m), s) = \prod_i L^S(\rho_i, s).$$

By (5b) this gives

$$\begin{aligned} -\mathrm{ord}_{s=1} L^S(\rho_\ell(m), s) &= -\sum_i \mathrm{ord}_{s=1} L^S(\rho_i, s) \\ &= \dim_{\overline{\mathbb{Q}}_\ell}(V_\ell(m) \otimes \overline{\mathbb{Q}}_\ell)^{\Gamma_k} \\ &= \dim_{\mathbb{Q}_\ell} V_\ell(m)^{\Gamma_k} \\ &= r^{(m)}_{\ell,k}. \end{aligned}$$

On the other hand, applying the Tate twist to ρ_ℓ has the effect of translation on its L-function, namely $L^S(\rho_\ell(m), s) = L^S(\rho_\ell, s+m)$. Therefore,

$$r^{(m)}_{\mathrm{an},k} = -\mathrm{ord}_{s=m+1} L^S(\rho_\ell, s) = -\mathrm{ord}_{s=1} L^S(\rho_\ell(m), s) = r^{(m)}_{\ell,k}.$$

Since we automatically have $r^{(m)}_{\mathrm{alg},k} \le r^{(m)}_{\ell,k}$, this completes the proof. □

6. Remarks on the analogous question for number fields

The formulation of the Tate Conjecture in §1 for the case of global function fields also makes sense when k is a number field, provided that the finite set of places S also includes the archimedean ones. One can then ask when the inequality $r^{(m)}_{\mathrm{alg},k} \le r^{(m)}_{\mathrm{an},k}$ is known to hold. In most cases where this is known to be true, such as some Shimura varieties for $m = 1$ [BR],[Kli],[MR] or Hilbert modular fourfolds for $m = 2$ [Ram], or certain $K3$ surfaces [SI], the full Tate Conjecture has actually been established.

If the Langlands conjectures for GL_n over number fields could be established, one could use the methods in this article to prove

$$r^{(m)}_{\mathrm{alg},k} \le r^{(m)}_{\ell,k} = r^{(m)}_{\mathrm{an},k}, \tag{6a}$$

since Theorem 3.2 holds, in fact, for all global fields. We remark, though, that this conjectural correspondence for number fields is not just a simple analogue of Theorem 4.1 and Corollary 4.2, due to the extra difficulties imposed by the places lying over ℓ and ∞. One notable difference is that one must restrict attention to so-called *algebraic* cuspidal representations [Clo]. In case $n = 1$, this corresponds to A. Weil's notion of an *idele class character of type* A_0 [Wei]. This is an idele class character χ such that, if v is archimedean, then $\chi_v(z) = z^{p_v}\bar{z}^{q_v}$; furthermore, we have $p_v + q_v = w$ for some $w \in \mathbb{Z}$ (the *weight* of χ) and all such v.

We note in passing that an idele class character of type A_0 with nonzero weight gives a counterexample to Lemma 4.3 in the number field case, since no nonzero power would be trivial at the archimedean places.

Unfortunately, as it currently stands, the representation ρ_ℓ is known to correspond to an algebraic cuspidal representation in only a handful of cases. Below we discuss one case where enough is known about ρ_ℓ to establish (6a). Recall that an abelian variety X over k is said to be potentially CM-type if we can find a commutative semisimple algebra Λ of dimension $2(\dim X)$ over $\mathbb{Q}$ and an isomorphism

$$\theta\colon \Lambda \xrightarrow{\sim} \mathrm{End}_{\bar{k}}(X) \otimes \mathbb{Q}.$$

Proposition 6.1. *Let X be abelian variety over the number field k which is potentially CM-type. Then*

$$r^{(m)}_{\mathrm{alg},k} \leq r^{(m)}_{\ell,k} = r^{(m)}_{\mathrm{an},k}$$

for all $0 \leq m \leq \dim X$.

Proof. Keeping the notation above, there is a finite Galois extension L/k such that all elements of

$$\theta(\Lambda) \cap \mathrm{End}_{\bar{k}}(X)$$

are rational over L, and thus the action of Γ_L on $H^1_{\text{ét}}(X \times_k \bar{k}, \mathbb{Q}_\ell)$ is abelian. Hence Γ_L acts via a direct sum of characters if we extend scalars to $\overline{\mathbb{Q}}_\ell$, and these characters are associated to idele class characters of type A_0 in the sense given in the proof of Corollary 4.2 [ST].

It is known that, as with any abelian variety, we have an isomorphism of Γ_k-modules

$$H^r_{\text{ét}}(X \times_k \bar{k}, \mathbb{Q}_\ell) \simeq \wedge^r H^1_{\text{ét}}(X \times_k \bar{k}, \mathbb{Q}_\ell)$$

(see [Mum], for instance). Therefore the action of Γ_L on $H^r_{\text{ét}}(X \times_k \bar{k}, \mathbb{Q}_\ell)$ is also associated to idele class characters of type A_0 after extension to $\overline{\mathbb{Q}}_\ell$.

We focus on the case $r = 2m$, letting $V_\ell = H^{2m}_{\text{ét}}(X \times_k \bar{k}, \mathbb{Q}_\ell)$ and $\rho_\ell(m)\colon \Gamma_k \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(m))$ as in §5. Then once again the semisimplification of the extension of $\rho_\ell(m)$ is a direct sum of irreducible $\overline{\mathbb{Q}}_\ell$-representations M_i of Γ_k:

$$(V_\ell(m) \otimes \overline{\mathbb{Q}}_\ell)^{\mathrm{ss}} = \bigoplus_i M_i.$$

As before, let ρ_i denote the $\overline{\mathbb{Q}}_\ell$-representation defined on M_i by $\rho_\ell(m)$.

The observations above say that $\rho_{i|_{\Gamma_L}}$ is a direct sum of characters associated to idele class characters of type A_0. Due to this condition, a result of H. Yoshida ([Yos], Theorem 1) gives a continuous finite-dimensional complex representation

$$r_i\colon W_k \to \mathrm{Aut}_{\mathbb{C}}(N_i)$$

of the Weil group W_k of k and a finite set of places S such that

$$L_v(r_i, s) = L_v(\rho_i, s)$$

for all $v \notin S$. Yoshida's construction guarantees that r_i is irreducible if and only if ρ_i is irreducible ([Yos], Theorems 1 and 2), so r_i is irreducible. We refer the reader to [Tat2] for the notions of Weil groups, their representations, and the associated L-functions, as well as for facts listed below; for a very complete discussion of these matters, see [Del1].

Following the same strategy as in the proof of Theorem 2.1, it suffices, for the completion of the proposition, to establish that

$$-\mathrm{ord}_{s=1} L^S(r_i, s) = \dim_{\mathbb{C}} N_i^{W_k}. \tag{6b}$$

We will do this by relating r_i to characters of the Weil group, which are just idele class characters, and then using the analogue of Corollary 3.3 for the L-functions of such characters.

First we use the existence of a finite extension E_i/k such that r_i is the induction of a primitive representation of W_{E_i}. (Here, primitive means that it is not induced from a smaller subgroup.) In fact, one knows that $r_i = \mathrm{Ind}_{E_i}^k(t_i \otimes \chi_i)$, where t_i is a representation of W_{E_i} of *Galois type* and χ_i is a character of W_{E_i}. Thus t_i is a representation of W_{E_i} pulled back via the surjection $W_{E_i} \to \mathrm{Gal}(\overline{k}/E_i)$, while χ_i is an idele class character by virtue of the isomorphism $\mathbb{A}_{E_i}^\times/E_i^\times \simeq W_{E_i}^{ab}$.

Next, Brauer's induction theorem says that

$$t_i \oplus \bigoplus_\alpha n_\alpha \mathrm{Ind}_{F_\alpha}^{E_i}(\psi_\alpha) \simeq \bigoplus_\beta n'_\beta \mathrm{Ind}_{F'_\beta}^{E_i}(\psi'_\beta)$$

for some finite extensions F_α/E_i, F'_β/E_i, (idele class) characters ψ_α, ψ'_β, and positive integers n_α, n'_β. (In other words, t_i is a finite virtual sum of inductions of characters in the Grothendieck group.) Since $\mathrm{Ind}(\psi) \otimes \chi_i \simeq \mathrm{Ind}(\psi \otimes \mathrm{Res}(\chi_i))$, we have

$$(t_i \otimes \chi_i) \oplus \bigoplus_\alpha n_\alpha \mathrm{Ind}_{F_\alpha}^{E_i}(\psi_\alpha \mathrm{Res}_{F_\alpha}(\chi_i)) \simeq \bigoplus_\beta n'_\beta \mathrm{Ind}_{F'_\beta}^{E_i}(\psi'_\beta \mathrm{Res}_{F'_\beta}(\chi_i)). \tag{6c}$$

From this we conclude two things.

The first is that

$$\begin{aligned} L^S(r_i, s) &= L^{S_i}(t_i \otimes \chi_i, s) \\ &= \prod_\alpha L^{S_\alpha}(\psi_\alpha \mathrm{Res}_{F_\alpha}(\chi_i), s)^{-n_\alpha} \prod_\beta L^{S'_\beta}(\psi'_\beta \mathrm{Res}_{F'_\beta}(\chi_i), s)^{n'_\beta}, \end{aligned}$$

where S_i (resp., S_α, S'_β) is the finite set of places in E_i (resp., F_α, F'_β) lying above those in S. Hence we have

$$\begin{aligned} -\mathrm{ord}_{s=1} L^S(r_i, s) = &\sum_\alpha n_\alpha \mathrm{ord}_{s=1} L^{S_\alpha}(\psi_\alpha \mathrm{Res}_{F_\alpha}(\chi_i), s) \\ &- \sum_\beta n'_\beta \mathrm{ord}_{s=1} L^{S'_\beta}(\psi'_\beta \mathrm{Res}_{F'_\beta}(\chi_i), s). \end{aligned} \tag{6d}$$

The L-functions on the right side of (6d) are of the form $L^{\Sigma}(\omega, s)$ for some idele-class character ω and finite set of places Σ, and for such L-functions we have

$$-\mathrm{ord}_{s=1} L^{\Sigma}(\omega, s) = \begin{cases} 1 & \text{if } \omega = 1 \\ 0 & \text{if } \omega \neq 1 \end{cases}.$$

Hence (6d) becomes

$$\begin{aligned} -\mathrm{ord}_{s=1} L^{S}(r_i, s) = {} & \sum_{\beta} n'_{\beta} \left\{ \begin{array}{ll} 1 & \text{if } \psi'_{\beta}\mathrm{Res}_{F'_{\beta}}(\chi_i) = 1 \\ 0 & \text{if } \psi'_{\beta}\mathrm{Res}_{F'_{\beta}}(\chi_i) \neq 1 \end{array} \right\} \\ & - \sum_{\alpha} n_{\alpha} \left\{ \begin{array}{ll} 1 & \text{if } \psi_{\alpha}\mathrm{Res}_{F_{\alpha}}(\chi_i) = 1 \\ 0 & \text{if } \psi_{\alpha}\mathrm{Res}_{F_{\alpha}}(\chi_i) \neq 1 \end{array} \right\}. \end{aligned} \tag{6e}$$

The second consequence of (6c) is that the dimension of the trivial representation in $t_i \otimes \chi_i$ (and in r_i by induction) is equal to

$$\sum_{\beta} n'_{\beta} \left\{ \begin{array}{ll} 1 & \text{if } \psi'_{\beta}\mathrm{Res}_{F'_{\beta}}(\chi_i) = 1 \\ 0 & \text{if } \psi'_{\beta}\mathrm{Res}_{F'_{\beta}}(\chi_i) \neq 1 \end{array} \right\} - \sum_{\alpha} n_{\alpha} \left\{ \begin{array}{ll} 1 & \text{if } \psi_{\alpha}\mathrm{Res}_{F_{\alpha}}(\chi_i) = 1 \\ 0 & \text{if } \psi_{\alpha}\mathrm{Res}_{F_{\alpha}}(\chi_i) \neq 1 \end{array} \right\},$$

since $\mathrm{Ind}(\omega)$ will contain the trivial representation only if $\omega = 1$, and in that case it will occur with dimension one. So putting this together with (6e), we get (6b) as desired. □

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Department of Mathematics 253-37, California Institute of Technology, Pasadena, California 91125, USA

E-mail address: clyons@caltech.edu